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Strength reliability of statically indeterminate heterogeneous beams

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Abstract

The strength reliability of linearly elastic (up to failure) beams, made from random heterogeneous microstructures is studied, based on the weakest link approach. Heterogeneity is confined to the longitudinal direction. The problem is statically indeterminate, and the local stress at each point in any cross section is a function of the stiffness morphology of the whole beam. External loading is not random, but reaction forces are, due to their statistical correlation with the beam morphology. The case of one degree of indeterminacy is studied here, for simplicity. The strength and reliability of the beam, being a stochastic function of local stresses, is therefore morphology dependent, in addition to (coupled with) the classical inherent probabilistic nature, associated with surface defects and irregularities. This dependence is found analytically as a function of external loading shape. A simple design formula for the bound of these effects on the beam strength has been found, covering any possible external loading. For example, for a beam of 10 grains (compliance correlation length of $0.1L$) and a 10% compliance variance, the bound of the heterogeneity effect on strength is about 8%.

Keywords: Strength; Reliability; Beam; Elasticity; Probability; Heterogeneity; Morphology; Bounds; Statically indeterminate

1. Introduction and motivation

Microsize beams (microbeams) are routinely used in MEMS applications such as sensors and actuators. These microstructures are usually linear elastic, brittle, and their reliability (related to strength), is strongly dependent on surface properties. In addition, their basic sub-element (grain) size is not negligible, when compared to some typical overall dimension (length, thickness), so the structure is not homogeneous. For example, in the case of microbeams made of polycrystals, material heterogeneity is expected to have a strong effect on strength due to local stress concentrations, caused by nonuniform deformations of neighboring grains. This is in spite of the extremely uniform dimensions and surface smoothness, which are obtained by MEMS technology. Therefore, the above heterogeneities can still be regarded as "equivalent surface defects" for the analysis of strength.

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Consider as a reference the problem of finding the strength reliability of a homogeneous Euler–Bernoulli beam, subjected to a nonrandom external loading. For statically determinate cases, stresses are independent of moduli. Failure is governed by surface defects and local stress concentrations, which are random. Therefore, if the failure probability of a unit length of the beam is given, the strength reliability of the whole beam is found by direct integration. This type of problem has been studied extensively both for static (Elishakoff, 1983) and dynamic (Lin and Cai, 1995) conditions.

Now let the beam be longitudinally heterogeneous, i.e., the material is uniform through the cross section but stochastically nonuniform along the length of the beam. For statically determinate cases, the stresses are still found merely from equilibrium considerations and are not correlated to moduli. Deflections can also be found directly (Koyluoglu et al., 1994; Beran, 1998; Elishakoff et al., 1999). As for the homogeneous case, the strength reliability of the whole beam can be calculated exactly.

Indeterminate beams pose new complications for reliability analysis. Local stresses are morphology dependent (moduli, grain size statistics etc.) through structural compatibility conditions. Moreover, reaction forces are also randomly coupled with heterogeneity. Therefore, three sources of random fields are nonlinearly involved: material heterogeneity, reaction forces and strength.

Approximations for the average reaction forces and their statistical variance for heterogeneous indeterminate beams have been recently found analytically (Altus, 2001). The results are used in the following, to study the strength reliability of statically indeterminate, isotropically heterogeneous beams, based on the weakest link approach. The main objective here is to assess the contribution of stiffness heterogeneity, which is caused by indeterminacy, on the strength reliability. In addition, the magnitude of this phenomenon is expected to depend on the "shape" of the external loading. Therefore, a proper bound for this effect, which covers all possible shapes, is desirable for design.

The probabilistic nature of the strength of polysilicon microstructures, has been experimentally observed (Jones et al., 1999; Greek et al., 1999), but the models used so far have not considered morphology effects.

The general problem of strength of heterogeneous media, including morphological effects and using more advanced failure criteria have been studied extensively (for example, Herrmann and Roux, 1990; Jeulin, 1993). However, the above compatibility boundary effects were not considered.

The paper is presented in the following order. Section 2 introduces basic relations of stochastic strength, from which the justification of a power law approximation for failure probability of a beam of unit length is established. In Section 3, a heterogeneous beam having one degree of indeterminacy is studied (clamped-simply supported), and the statistical average and variance of the reaction force is found. These results are used in Section 4, to analytically find the strength reliability of a beam under any specific (nonrandom) loading. An upper bound is then proposed for any loading shape. A discussion on the quality of this bound is given in Section 5 with some loading examples for comparison.

2. Basic relations of random strength

2.1. Consequences of the weakest link approach

Consider a 1 D rod under a concentrated (P) and distributed ($q(x)$) loads as shown in Fig. 1. The strength of the rod is controlled by surface defects, microstructure heterogeneity etc., (i.e., random). Fracture mechanics tools are difficult to apply, since defect geometries and stress concentration fields are too complex to be measured quantitatively. Therefore, in order to find the strength reliability of the rod, we have to rely on some reference (experimental) data, which corresponds to the failure probability of an element of a standard (but arbitrarily chosen) length ℓ, subjected to a uniform stress $\bar{\sigma}$, i.e.,

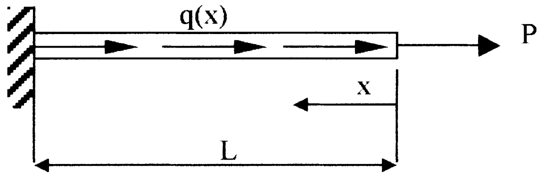

Fig. 1. One dimensional heterogeneous rod, loaded by $q(x)$ and P.

$$F(\bar{\sigma},\ell) = \int_0^{\bar{\sigma}} f(\sigma',\ell)\,\mathrm{d}\sigma' = 1 - G(\bar{\sigma},\ell), \tag{2.1}$$

where F and G are failure and survival probabilities, respectively, and f is the failure probability density. The problem now is to express the failure probability of the whole rod as a function of the properties of the standard element.

Divide the rod of the total length L into n equal elements of length $\ell = L/n$. Apply the weakest link approach for strength and assume that the failure probabilities of the elements are not correlated. Then,

$$G(\sigma(x),L) = \prod_{i=1}^{n} G(\bar{\sigma}_i,\ell), \tag{2.2}$$

where now G is a functional of $\sigma(x)$. We take that ℓ is small enough (n large), such that a uniform stress ($\bar{\sigma}_i$) inside each basic element (i) is assumed. For any G, we can define (Weibull, 1951):

$$g(\bar{\sigma},\ell) = -\ln G(\bar{\sigma},\ell) \to F(\bar{\sigma},\ell) = 1 - \exp(-g(\bar{\sigma},\ell)). \tag{2.3}$$

From (2.3), (2.2) can be rewritten as:

$$g(\sigma(x),L) = \sum_{i=1}^{n} g(\bar{\sigma}_i,\ell). \tag{2.4}$$

This additive property of g will be used in the following.

Consider now a special case, for which $q(x) = 0$ and the stress field along the rod is uniform. Given L, the survival probability of the whole rod is a physical (measurable) quantity, which must be independent of n. Therefore, $g(\bar{\sigma},L)$ is invariant with the size of ℓ. Using (2.4) for two arbitrary lengths ℓ_1 and ℓ_2 yields:

$$g(\bar{\sigma},L) = n_1 g(\bar{\sigma},\ell_1) = n_2 g(\bar{\sigma},\ell_2); \quad n_1\ell_1 = n_2\ell_2 = L. \tag{2.5}$$

Combining (2.5a) and (b) we have:

$$\ell_2 g(\bar{\sigma},\ell_1) = \ell_1 g(\bar{\sigma},\ell_2). \tag{2.6}$$

Now define

$$g_0(\bar{\sigma}) = \frac{g(\bar{\sigma},\ell_1)}{\ell_1} = \frac{g(\bar{\sigma},\ell_2)}{\ell_2} = g(\bar{\sigma},1) \tag{2.7}$$

which is a material property independent of length. Therefore,

$$g(\bar{\sigma},\ell) = \ell \cdot g_0(\bar{\sigma}). \tag{2.8}$$

Substituting in (2.4) we have:

$$g(\sigma(x),L) = \sum_{i=1}^{n} g(\bar{\sigma}_i,\ell) = \sum_{i=1}^{n} g_0(\bar{\sigma}_i)\cdot\ell, \tag{2.9}$$

which, in a continuous form leads simply to:

$$g(\sigma(x),L)=\int_0^L g_0(\sigma(x))\,\mathrm{d}x=\frac{1}{\ell}\int_0^L g(\sigma(x),\ell)\,\mathrm{d}x=\frac{L}{\ell}\int_0^1 g(\sigma(x),\ell)\,\mathrm{d}x, \tag{2.10}$$

where x in the last term is normalized by L. The above is a form of the very well known dependence of the survival probability of a structure on the stress field, used frequently for ceramics (Davidge, 1979). Interestingly, from (2.10) we obtain that when $L\to 0$, $g\to 0$ and $G\to 1$, which means that a vanishingly small element never fails. However, this limit possesses no practical difficulty, since measurable elements have always a finite size and a finite strength. This singularity feature resembles the theoretical linear elastic stress field near cracks, where infinite stresses correspond to a zero crack tip radius, although the elastic energy in a finite volume near the tip is bounded.

We see that (2.10) provides a convenient representation of the survival probability of the rod, in terms of its unit strength ("material") property. Similar to statistical thermodynamics, where entropy is defined by a quantity derived from a logarithm of probabilities, g can be conceived by (2.3) as an "entropy of strength".

2.2. Weibull distribution function and the power law approximation

Let us confine our study to loads of low failure probability, which is very practical in MEMS, where high reliability is in demand. Then, the above properties of g can be used for two very useful approximations, which will help us in achieving analytical results in the following. First, from (2.3) we have (Ashby and Jones, 1986):

$$g(\bar{\sigma},\ell)\cong F(\bar{\sigma},\ell). \tag{2.11}$$

Secondly, high reliability means low stresses (relative to the average failure stress). Therefore g can be approximated in the range of low stresses as a power function:

$$g(\bar{\sigma},\ell)=\left(\frac{\bar{\sigma}}{\sigma_\ell}\right)^\beta. \tag{2.12}$$

Using (2.12) and inserting in (2.3), we get the Weibull (1951) distribution function, extensively used for failure probability analysis of brittle materials:

$$F_W(\bar{\sigma},\ell)=1-\exp\left(-\left(\frac{\bar{\sigma}}{\sigma_\ell}\right)^\beta\right), \tag{2.13}$$

where σ_ℓ is approximately the ensemble average strength and β is the material shape parameter. We see that F_W is a good approximation for a large class of distributions, as long as the stresses are low enough. When the information for high stresses is important too, it may fit only special cases.

When both (2.11) and (2.12) are used, we have simply:

$$F(\sigma,\ell)\cong g(\bar{\sigma},\ell)=\left(\frac{\bar{\sigma}}{\sigma_\ell}\right)^\beta. \tag{2.14}$$

We see that not only that the power form is very convenient for analytical manipulations, it is a good approximation for any distribution in the low stress range *and is not* confined to the Weibull distribution.

As a demonstration for the power law accuracy and capacity, consider a rod of a length of 10 basic elements ($L/\ell=10$), having a Weibull failure probability density (associated with ℓ) for which the shape parameter β equals to 4. From (2.1) and (2.2), the failure probability of the whole rod is:

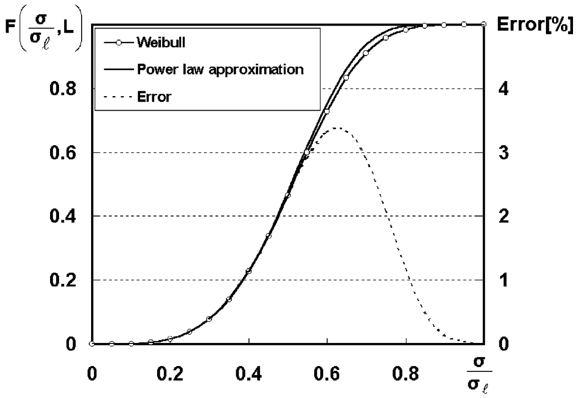

Fig. 2. Failure probability of a rod of length L. Comparison between data based on a Weibull distribution function, and a power law approximation, both for elements of length $L/\ell = 10$, and $\beta = 4$.

$$F(\bar{\sigma}, L) = 1 - [1 - F(\bar{\sigma}, \ell)]^{L/\ell}. \tag{2.15}$$

Now we calculate (2.15) directly by using F_W (2.13) and by the power approximation (2.14). Results are shown in Fig. 2 and the relative difference is also plotted for convenience. It is seen that the maximum error is in the order of 3%. Higher β and/or n show even smaller errors. It is therefore reasonable to assume, that the power law approximation for the failure probability of a standard element is expected to produce good predictions for many other failure probability functions (not just Weibull), as long as L/ℓ is large enough. It should be noted also that for $\bar{\sigma}/\sigma_\ell > 1$, the power law results are erroneous (failure probability greater than 1), but this stress region is not of interest here.

Finally, since F_W is commonly used for brittle materials, it is important to note that the Weibull distribution possesses various properties as related to its pdf gradient ($f_{,\sigma}$) at $\sigma = 0$ for different β values, i.e.,

$$f_{,\sigma}(0) = F_{,\sigma\sigma}(0) = \begin{bmatrix} = 0; & (\beta > 2) \\ > 0; & (\beta = 2) \\ = +\infty; & (1 < \beta < 2) \\ > -1; & (\beta = 1) \end{bmatrix}. \tag{2.16}$$

Therefore, practical values of β are expected to be greater than 2, a conclusion which will be used in the following. Indeed, it is found that $5 < \beta < 25$ for most brittle materials (Davidge, 1979; Elbrecht and Binder, 1999; Greek et al., 1999; Jones et al., 1999; Sharpe et al., 1999).

3. Average and variance of the indeterminate reaction force

Consider a clamped-simply supported beam (indeterminacy of degree 1), shown schematically in Fig. 3. The internal bending moment is:

$$M(x) = M_R + M_q; \quad M_R = R \cdot x; \quad M_q = \int_{x'=0}^{x} q(x - x') \cdot x' \, dx', \tag{3.1}$$

R is the reaction force at $x = 0$, chosen to be the "indeterminate parameter". $M_R(x)$ and $M_q(x)$ are the internal bending moment distributions caused separately by R and $q(x)$, respectively. R is found by the compatibility condition:

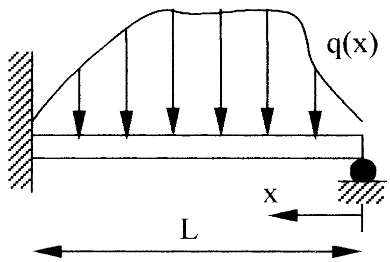

Fig. 3. A beam with one degree of indeterminacy.

$$\frac{\partial}{\partial R}\int_0^L \frac{1}{2}M^2 S(x)\,\mathrm{d}x = \int_0^L MxS(x)\,\mathrm{d}x = 0, \tag{3.2}$$

where (3.1) is used and $S(x)$ is the cross sectional bending compliance, i.e.,

$$S(x) = [EI(x)]^{-1} \tag{3.3}$$

E is Young's modulus and I is the appropriate cross sectional inertia term.

For a homogeneous beam, define:

$$S(x) = S_h; \quad R_h = R(S_h); \quad M_{R_h}(x) = R_h x; \quad M_h = M_q + M_{R_h}. \tag{3.4}$$

Then, applying (3.2) we have

$$\int_0^L M_h x\,\mathrm{d}x = \int_0^L (M_{R_h} + M_q)x\,\mathrm{d}x = 0, \tag{3.5}$$

so that

$$R_h = \frac{3}{L^3}\int_0^L M_q x\,\mathrm{d}x, \tag{3.6}$$

which is independent of S_h.

Before continuing, we define the convolution notation ($*$), which will be used throughout for convenience. If we have two, single variable functions $u(x)$ and $v(x)$, then,

$$u * v = \int u(x)v(x)\,\mathrm{d}x, \tag{3.7}$$

where the limits of integration are defined explicitly for each case. When both u and v are two variable functions, we define similarly:

$$u * v = \int u(x_1, x_2)v(x_2, x_3)\,\mathrm{d}x_2. \tag{3.8}$$

Therefore, ($*$) act on integration the same way inner products acts on tensors, i.e., if S, u, v are three functions of a single variable x, the notation

$$uv = u(x_1)v(x_2) \tag{3.9}$$

is defined as an outer functional product. Using the above, we write

$$u * (SS) * v = \int\int u(x_1)[S(x_1)S(x_2)]v(x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2, \tag{3.10}$$

where (SS) is an outer functional product. In fact, $(*)$ is treated here as an inner product between tensors or vectors of infinite terms in the appropriate dimension.

Now we write (3.2) in a normalized form, using the above convolution notations:

$$(Mx) * S = 0; \quad M \to \frac{M}{R_h L}, \quad S \to \frac{S}{\langle S \rangle}, \quad x \to \frac{x}{L}, \quad R \to \frac{R}{R_h}, \quad \langle S \rangle \to 1 \tag{3.11}$$

which will also be maintained throughout. The beam is considered as statistically homogeneous, therefore $\langle S \rangle$ stands for both the spatial and ensemble averages, and is not a function of x. The form in (3.11a) is identical to all beam problems having "one degree" indeterminacy. Using (3.1)–(3.4), (3.11) and the convolution notation, we obtain an expression for the reaction force for a heterogeneous beam:

$$R = -\frac{[M_q x] * S}{[M_{R_h} x] * S} = -\frac{[M_q x] * S}{x^2 * S}. \tag{3.12}$$

To find the survival probability (reliability) for each realization (i.e., $S(x)$ is given), we find R from (3.12), insert in (3.1), find the stresses along the beam and use (2.10) and (2.14) to obtain its failure probability. However, it is not practical to measure S of every beam, so we rely on its statistical (morphological) information, usually the n points correlation functions such as $\langle S(x) \rangle$, $\langle S'(x_1) S'(x_2) \rangle$, $\langle S'(x_1) S'(x_2) S'(x_3) \rangle$ etc. Using this partial information of heterogeneity, we intend to find partial (statistical) data on the beam strength (average failure load, variance and reliability). To achieve this goal, we have to find the relevant statistical information of R, in this case $\langle R \rangle$ and $\langle R'^2 \rangle$.

To obtain $\langle R \rangle$, we notice from (3.12) that it is a nonlinear functional of $S(x)$ and can be found analytically only by successive approximations. Using an expansion near $S = \langle S \rangle = 1$ we have

$$R = R|_{\langle S \rangle} + R_{,S}|_{\langle S \rangle} * S' + \tfrac{1}{2} R_{,SS}|_{\langle S \rangle} ** (S'S') + \cdots; \quad S' = S - 1, \tag{3.13}$$

where functional differentiations (variations) are used. Recalling the resemblance of our convolutions with tensor operations, we see that the variations in (3.13) are always "external" in the following sense:

$$R_{,SS_{\langle S \rangle}}(x, X) \equiv \left. \frac{\delta^2 R}{\delta S(x) \delta S(X)} \right|_{S = \langle S \rangle} = \left. \frac{\delta^2 R}{\delta S'(x) \delta S'(X)} \right|_{S' = 0}. \tag{3.14}$$

Averaging (3.12), the second term vanishes identically. Using (3.14) and inserting (3.12) in (3.13) we obtain:

$$\langle R \rangle = 1 + \Delta R_h \cong 1 + \frac{(M_h x) * \langle S'(x) \cdot S'(X) \rangle * (X^2)}{(x^2 * 1)^2}, \tag{3.15}$$

where $M_h(x)$ is the internal moment for the indeterminate homogeneous beam (a function of loading but morphology independent). We see that the average reaction force differs from the one for the homogeneous case by a term, which depends on the microstructure (two-point compliance correlation) *and* the loading shape.

Eq. (3.15) provides a solution for any given morphology, through $\langle S'S' \rangle$. However, more than interested in finding a particular solution for a specific correlation function, we prefer to find some general features, which correspond to certain morphological classes. Therefore, we examine the cases where the statistical correlation length is relatively small, i.e., $\langle S'S' \rangle$ contributes to the integral in (3.15) (i.e., nonzero) only when $|x - X|$ is small enough. Then, it is convenient to write (3.15) in the form

$$\Delta R_h \cong \frac{(M_h x) * \langle S'(x) \cdot S'(x + \varepsilon) \rangle * (x + \varepsilon)^2}{(x^2 * 1)^2}; \quad \varepsilon = X - x, \tag{3.16}$$

where now the convolution on the right side of the numerator is on ε. Since $\langle S'S' \rangle$ is a symmetric function of ε and independent of x (statistically homogeneous material) we have

$$\Delta R_h \cong \frac{(M_h x) * (x^2 \langle S'S' \rangle * 1 + 2x\langle S'S' \rangle * \varepsilon + \langle S'S' \rangle * \varepsilon^2)}{(x^2 * 1)^2}. \tag{3.17}$$

The second term in the numerator vanishes identically. The third term is two orders of ε smaller than the first, so (3.17) is simplified to:

$$\langle R \rangle \cong 1 + \frac{((M_h x) * (x^2)) \cdot \langle S' * S' \rangle}{(x^2 * 1)^2}. \tag{3.18}$$

Finally, a morphological correlation length (λ) can be defined as:

$$\lambda = \frac{\langle S' * S' \rangle}{\langle S'^2 \rangle} \tag{3.19}$$

and therefore (3.18) can be simplified further to:

$$\langle R \rangle = 1 + \lambda \langle S'^2 \rangle \cdot \frac{M_h * x^3}{(x^2 * 1)^2} = 1 + \lambda \langle S'^2 \rangle \cdot 9 M_h * x^3. \tag{3.20}$$

λ is associated with the typical grain size, for cases when there is no moduli correlation between adjacent grains. More details on the above (although in different terms), including the loading shape effect can be found elsewhere (Altus, 2001). We see that the deviation of $\langle R \rangle$ from R_h is proportional to the grain size, stiffness variance and is a function of the loading shape.

By the same procedures (details are not given here), the reaction force variance can also be found:

$$\langle R'^2 \rangle = \frac{(M_h x) * \langle S'S' \rangle * (M_h X)}{(x^2 * 1)^2} \cong \lambda \langle S'^2 \rangle \cdot \frac{M_h^2 * x^2}{(x^2 * 1)^2} = \lambda \langle S'^2 \rangle \cdot 9 M_h^2 * x^2. \tag{3.21}$$

Both (3.20) and (3.21) will be used in the following.

4. Strength reliability of an indeterminate beam

Using (2.10), (2.12) and (2.14), the failure probability of a beam, which is loaded by a distributed load $q(x)$ and an additional (arbitrary, nonrandom) external force R is:

$$F_R = \frac{L}{\ell} \int_0^1 (\sigma(x, R))^\beta \, \mathrm{d}x, \tag{4.1}$$

where x is the normalized length and σ is the normalized (by σ_ℓ) stress. From elementary Euler–Bernoulli analysis we have:

$$\sigma(x, R) = B \cdot M(x, R), \tag{4.2}$$

where B is a factor, related to the beam cross section geometry. We consider here failure due to near surface stresses only, appropriate for brittle materials subjected to bending. For simplicity, we take symmetric cross section geometry, where the maximum and minimum stresses are found at equal distances, but on opposite sides from the cross sectional center of gravity. Inserting (4.2) in (4.1) yields:

$$F_R = \frac{L}{\ell} B^\beta \int_0^1 |M(x, R)|^\beta \, \mathrm{d}x. \tag{4.3}$$

Recall that (4.3) is the failure probability of a *determinate* beam, with nonrandom loadings $q(x)$ and R. Considering now the indeterminate case, R is a random variable which is correlated to the heterogeneity.

The failure probability for the whole statistical ensemble of beams of different R values, can now be written as:

$$F_b = \int_{-\infty}^{\infty} F_R \cdot p_R \, \mathrm{d}R, \tag{4.4}$$

where $p_R(R)$ is the probability (ensemble) density function of R. Eq. (4.4) may be interpreted as the summation over failure probabilities of each possible value (range) of R, multiplied by the probability of having R in this range. In other words, F_b is just the statistical average of the failure probability, with respect to R:

$$F_b = \langle F_R(R) \rangle. \tag{4.5}$$

Expansion of F_R near $\langle R \rangle$ and averaging yields:

$$F_b = F_R|_{\langle R \rangle} + \tfrac{1}{2} F_{R,RR}|_{\langle R \rangle} \cdot \langle R'^2 \rangle + \cdots. \tag{4.6}$$

Since $\langle R \rangle$ is unknown (in the sense that it is not given as part of the data, or can be calculated exactly), we have to approximate F_b with association to R_h. From (3.18) and (3.21) we have that ΔR_h $(= \langle R \rangle - R_h)$ and $\langle R'^2 \rangle$ are both in the order of $\langle S'^2 \rangle$. Therefore, if

$$F_R|_{\langle R \rangle} = F_R|_{R_h} + F_{R,R}|_{R_h} \cdot \Delta R_h + \cdots \tag{4.7}$$

then the second term in (4.7) is in the same order (with respect to $\langle S'^2 \rangle$) as the second term in (4.6). Thus, (4.6) can be also approximated as:

$$F_b = F_R|_{R_h} + F_{R,R}|_{R_h} \cdot \Delta R_h + \tfrac{1}{2} F_{R,RR}|_{R_h} \cdot \langle R'^2 \rangle + \mathrm{O}(R'^3), \tag{4.8}$$

where the first term is the zero order (homogeneous) approximation and the other two terms reflect the contribution of stiffness heterogeneity to the overall failure probability. Calculating the coefficient of the third term in (4.8) at R_h instead of $\langle R \rangle$ is permitted since the difference between the two is in the order smaller than $\langle S'S'S' \rangle$.

Our main goal is to study the relative heterogeneity effect on F_b, i.e., the ratio between the sum of the second and third term in (4.8) to the first. To find it, note first that from (3.1) we have:

$$M = M_q + xR \to \begin{matrix} M > 0 \to |M| = M_q + xR \\ M < 0 \to |M| = -(M_q + xR) \end{matrix}. \tag{4.9}$$

Therefore,

$$|M|_{,R} = \begin{bmatrix} x; & M > 0 \\ -x; & M < 0 \end{bmatrix} = \mathrm{sign}(M(x)) \cdot x. \tag{4.10}$$

Using (4.10) and (4.3) we obtain:

$$F_{R,R} = \beta \Big[|M|^{\beta-1} * x|_{M>0} - |M|^{\beta-1} * x|_{M<0} \Big] = \beta |M|^{\beta-1} * (\mathrm{sign}(M) \cdot x), \tag{4.11}$$

which can be either positive or negative, depending on the sign of M along x. To calculate the second derivative ($F_{,RR}$), we first show that $(\mathrm{sign}(M))_{,R} = 0$:

$$\begin{aligned}[\mathrm{sign}(M)]_{,R} &= (M \cdot |M|^{-1})_{,R} = M_{,R}|M|^{-1} - M|M|^{-2}|M|_{,R} = |M|^{-2}[M_{,R}|M| - M|M|_{,R}] \\ &= |M|^{-2}[x \cdot |M| - M \cdot \mathrm{sign}(M) \cdot x] = |M|^{-2}[x \cdot |M| - |M| \cdot x] = 0. \end{aligned} \tag{4.12}$$

Then, differentiating (4.11) yields

$$F_{R,RR} = \beta(\beta - 1) \cdot |M|^{\beta-2} (\mathrm{sign}(M) \cdot x) * (\mathrm{sign}(M) \cdot x) = \beta(\beta - 1) \cdot |M|^{\beta-2} * x^2 \tag{4.13}$$

which is always positive.

Returning to (4.8), it can be shown that although ΔR_h and $\langle R'^2\rangle$ are of order $\langle S'S'\rangle$, the second term is expected to be much smaller than the third. To see this, we use (3.20), (3.21), (4.10), and (4.11), and find the ratio between the third and the second terms of (4.8) as:

$$r = \frac{F_{b2}}{F_{b3}} = \frac{2}{(\beta-1)} \frac{\left[|M_h|^{\beta-1} * (\mathrm{sign}(M_h)\cdot x)\right][M_h * x^3]}{[|M_h|^{\beta-2} * x^2][M_h^2 * x^2]}. \tag{4.14}$$

From (3.5) we obtain that when $\beta = 2$, the LHS square brackets of the numerator in (4.14) vanishes identically. Moreover, (3.5) shows that $M_h(x)$ must have both positive and negative parts, so that both integrands of the convolutions of the numerator have two parts which partially cancel each other, while in the denominator, the integrands are positive along the whole integration path. Besides, r tends to $2/\beta$ for high enough β values.

We therefore take that $r \ll 1$ for the entire range of β, neglect the second term in (4.8) and write the effect of the stiffness heterogeneity on the strength reliability of the beam as the ratio (η) between the third and the first terms only. Using (4.3), (4.8) and (4.13) we obtain:

$$\eta = \frac{1}{2}\frac{F_{R,RR}|_{R_h}\cdot\langle R'^2\rangle}{F_R|_{R_h}} = \lambda\langle S'^2\rangle\beta(\beta-1)\cdot\frac{9}{2}\frac{\eta_1\cdot\eta_2}{\eta_3}, \tag{4.15}$$

where

$$\eta_1 = |M_h|^{\beta-2} * x^2; \quad \eta_2 = M_h^2 * x^2; \quad \eta_3 = |M_h|^{\beta} * 1. \tag{4.16}$$

Therefore, η is a function of $|M_h|$ only. The distinction made in (4.15) between $\beta(\beta-1)$ and the other factors stems from the fact that this factor is *not* a function of loading (as η_j), but depends on the power law strength distribution and the double differentiation from the Taylor expansion. Thus, (4.15) shows explicitly the effect of various properties, coming from different sources on η: microstructure by λ, material inhomogeneity by $\langle S'^2\rangle$, strength randomness by β, and loading shape by M_h.

Although (4.15) is explicit and straightforward to calculate for any given M_h (through $q(x)$ and β), it is also desirable for design purposes, to study what are those loading shapes for which η receives its extreme values, or alternatively, find some bounds (if such bounds exist) for *any* possible loading ($q(x)$). Instead of writing the direct relation between η and $q(x)$, we use the fact that η is an explicit functional of $|M_h|$, and look for $|M_h|$ for which η is maximal. Then, we find the associated loading $q(x)$ through (3.1) and (3.5). From (4.15) it is clear that η is insensitive to the *magnitude* of M_h. Therefore, it is convenient to choose M_h such that η_3 is equal to an arbitrary constant C_3. Then, the two conditions:

$$\frac{\delta\eta}{\delta|M_h|} = 0 \leftrightarrow \frac{\delta}{\delta|M_h|}[(\eta_1\eta_2) + \omega(C_3 - \eta_3)] = 0, \tag{4.17}$$

where ω is a Lagrange multiplier, are equivalent. Inserting (4.16) into (4.17b) and taking the variation with respect to $|M_h|$, we obtain an equation of the form:

$$a_1|M_h|^{\beta-3}x^2 + a_2|M_h|x^2 + a_3|M_h|^{\beta-1} = 0, \tag{4.18}$$

where a_1, a_2, a_3 are constants, i.e., not functions of x, even though a_1, a_2, a_3 are convolutions of the unknown moment function:

$$a_1 = (\beta-2)|M_h|^2 * x^2; \quad a_2 = 2|M_h|^{\beta-2} * x^2; \quad a_3 = \omega\beta C_3. \tag{4.19}$$

Eq. (4.18) has an exact solution when $\beta = 4$, for which $\eta_1 = \eta_2$, in the form

$$\beta = 4 \rightarrow |M_h| = Cx \rightarrow \eta = 10.8\cdot\lambda\langle S'^2\rangle, \tag{4.20}$$

while for other β values a simple power law solution is impossible (other analytic solutions could not be found). Therefore, we use the fact that η_1 and η_2 are always positive to write:

$$\eta^{\max} \leqslant \eta_1^{\max}\eta_2^{\max} = \eta^+, \tag{4.21}$$

where η^+ is an upper bound for η. Using the same procedure as in (4.17b) for η_1 and η_2 separately we obtain:

$$a_1|M_h|^{\beta-3}x^2 + a_3|M_h|^{\beta-1} = 0 \rightarrow |M_h|_1^+ = C_1x, \tag{4.22}$$

$$a_2|M_h|x^2 + a_3|M_h|^{\beta-1} = 0 \rightarrow |M_h|_2^+ = C_2x^{2/(\beta-2)}, \tag{4.23}$$

C_1 and C_2 are constants, calculated by the additional condition $\eta_3 = C_3$ for each case. Inserting (4.22) and (4.23) in (4.16) and (4.15) and integrating yields:

$$\eta^+ = \lambda\langle S'^2\rangle \cdot \beta(\beta-1) \cdot \frac{9}{2}\left(\frac{\beta-2}{3\beta-2}\right)^{1-2/\beta} \cdot (\beta+1)^{-2/\beta}. \tag{4.24}$$

Note that although η^+ is an upper bound for the effect of morphology on strength, the moment distributions in (4.22) and (4.23) are *not* realizations of $M(x)$ for such bounds, except for $\beta = 4$, when $|M_{h1}| = |M_{h2}| = Cx$. In this special case, the bound is a real supremum. Since $\beta > 2$ for common materials, the singular point ($\beta = 2/3$) in (4.24) is not physical. Moreover, it is easy to see that for ($\beta > 5$) values, (4.24) yields:

$$\eta^+_{(\beta>5)} < (3/2) \cdot \lambda\langle S'^2\rangle \cdot \beta(\beta-1), \tag{4.25}$$

which can be proposed as a convenient design formula. For completeness, it should be noted that the factors (1/2) and (3) in (4.25) come from the second order approximation of the Taylor expansion and the linear relation between M_R and R, respectively.

The practical significance of (4.25) can be demonstrated by taking an example of a beam for which $\lambda\langle S'^2\rangle$ is in the order of $\sim$0.01, $10 < \beta < 15$, which, by (4.25) leads to $0.92 < \eta^+ < 2.3$. This means that the contribution of the stiffness heterogeneity to the failure probability of the beam can be more important ($\eta^+ > 1$) than the one caused by the classical strength randomness (i.e., homogeneous beam).

We can "transfer" the stiffness heterogeneity effect to the loading space, i.e., asking how much the magnitude of the allowable (design) external loads changes for a given failure probability, due to the fact that the material is not homogeneous. From (4.3) and (4.6) we have:

$$F_b \cong F_R|_{\langle R\rangle} + \tfrac{1}{2}F_{R,RR}|_{\langle R\rangle} \cdot \langle R'^2\rangle = (k_1 + k_2)q_0^\beta, \tag{4.26}$$

where q_0 is the loading magnitude for a given failure probability F_b, and k_1, k_2 are the relative contributions due to the strength randomness and moduli heterogeneity, respectively. From (4.15) we have,

$$\left(\frac{k_2}{k_1}\right)^+ = \eta^+. \tag{4.27}$$

Now, we compare two calculations for the same problem, with and without the heterogeneity effect, for the same reliability prediction, i.e.,

$$F_b^{\text{hom}} = k_1(q_0^{\text{hom}})^\beta; \quad F_b^{\text{het}} = (k_1 + k_2)(q_0^{\text{het}})^\beta; \quad F_b^{\text{hom}} = F_b^{\text{het}}. \tag{4.28}$$

Then, (4.27) and (4.28) yields:

$$\Delta q_0 = 1 - \frac{q_0^{\text{het}}}{q_0^{\text{hom}}} = 1 - \left(1 + \frac{k_2}{k_1}\right)^{-1/\beta} = 1 - (1 + \eta^+)^{-1/\beta}. \tag{4.29}$$

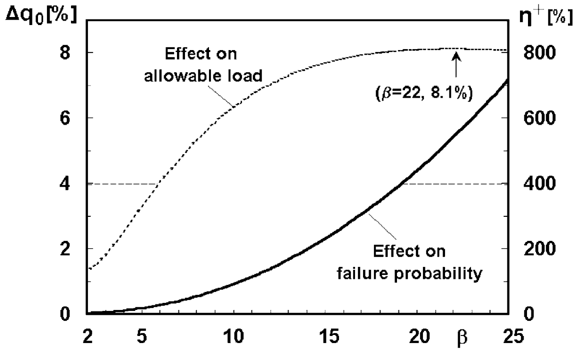

Fig. 4. Effect of stiffness heterogeneity on the beam strength reliability. $\lambda\langle S'^2\rangle = 0.01$.

Taking the same values for the practical example as above, Eq. (4.25) predicts that a maximal (upper bound) reduction of 8.2% from the classical design load should be added (i.e., safety factor of 1.09) for all materials ($\beta > 2$) and $\lambda\langle S'^2\rangle \sim 0.01$.

Both bound representations ((4.24) and (4.29)) are shown in Fig. 4 for the whole practical range $2 < \beta < 25$. We see that the effect on reliability is monotonic, while the effect on the design load has the above maximum at $\beta \approx 22$, a value within the range for brittle materials.

The nonmonotonic behavior of (4.29) reflects a complex interaction between two stochastic sources, assumed uncorrelated in this study: moduli morphology (reflected here by $\lambda\langle S'^2\rangle$) and strength ($\beta$). For example, a very high β value means that the material is relatively homogeneous with respect to strength. Therefore, failure of the beam is expected to occur at the point of maximum stress, i.e., the loading *shape* is immaterial, which means *weak* heterogeneity effect. On the other hand, since $\langle R'^2\rangle$ depends solely on moduli heterogeneity, the maximum stresses will still be stochastic, and since β is high, the sensitivity of the stress at this specific point to the beam failure is high too, inducing a *strong* stochastic strength effect.

5. Discussion

As for any upper bound, its quality is evaluated by finding how close this bound is to the supremum. Lacking analytical solutions for those loading shapes, which yield a supremum for the strength reliability, we find the heterogeneity effect for a few special cases of load distributions and examine how close their reliability are to the predicted bound. Results for four loading shapes, explained in the following, are compared to the upper bound predictions, as shown in Fig. 5.

The first example is a power law distribution for $|M_h|$, motivated by the analytical solution as in (4.22) and (4.23), i.e.,

$$|M_h| = A \cdot x^k. \tag{5.1}$$

Here we look for the maximal η for all possible k (>0).

Before proceeding further, notice that from mathematical convenience, the solution (and bound) is found in terms of $|M_h|$ and not as a direct function of external loads. To find $q(x)$ explicitly from (5.1), we first use (3.5) and obtain a solution of the form:

$$M_h = A \cdot x^k(1 - 2H(x - x_0)), \tag{5.2}$$

where $H(x - x_0)$ is a step (Heaviside) function located at x_0. Note that for a given $|M_h|$ and the condition (3.5), M_h is *not* uniquely defined, since one can find other functions with multiple steps which satisfy both (5.1) and (3.5). For the case of a single step, described by (5.2), we use (3.5) and (5.2) and obtain:

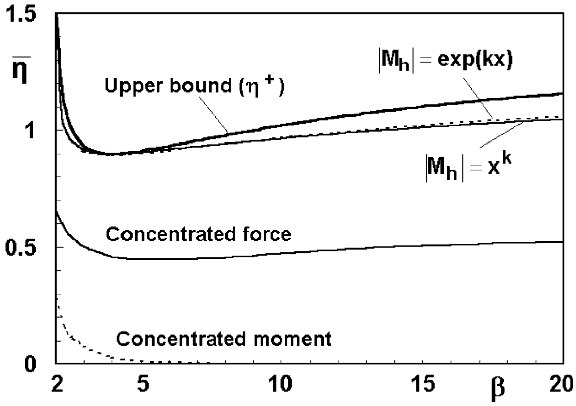

Fig. 5. Effect of various loading realizations on $\bar{\eta}$ as a function of β. The quality of the upper bound is measured by the smallest difference with any possible realization. $\lambda\langle S'^2\rangle = 0.01$.

$$x_0 = (2)^{-1/(k+2)}. \tag{5.3}$$

The external loading function ($q(x)$) is found by differentiation:

$$q(x) = M_{h,xx} = A \cdot kx^{k-2}\lfloor (k-1)(1-2H) - 4\delta x + 2\delta^2 x^2 \rfloor, \tag{5.4}$$

where $\delta(x - x_0)$ and δ^2 are the first and second order Dirac functions. The first term in (5.4) is a power law distribution with a (skew symmetric) step function, the second is a concentrated force and the third is a concentrated bending moment. The location of the step, force and moment is at $x = x_0$. $q(x)$ from (5.4) is shown schematically in Fig. 6. Note also that (5.4) represents nonphysical singularities at $x = 0$ for $k < 2$, but as will be shown, this problem is not of major importance here.

Inserting (5.1) in (4.15) we obtain

$$\bar{\eta} = \frac{\eta}{\lambda\langle S'^2\rangle \cdot \beta(\beta-1)} = \frac{9}{2}\frac{\eta_1\eta_2}{\eta_3} = \frac{9}{2}\frac{k\beta+1}{(k\beta-2k+3)(2k+3)}, \tag{5.5}$$

where $\bar{\eta}$ is the net loading shape effect. From (5.5), the value of k, for which η is maximal ($\eta_{\max}$) is:

$$k_0 = k_{(\bar{\eta}_{\max})} = \frac{1}{\beta}\left[\frac{(3\beta^3 - 5\beta^2 - 4\beta + 4)^{1/2}}{(\beta-2)} - 1\right]. \tag{5.6}$$

Fig. 5 shows $\bar{\eta}_{\max}$ after inserting (5.6) in (5.5). It is seen that $\bar{\eta}_{\max}$ is very close to the bound and equals the bound at $\beta = 4$, as expected. However, as seen from (5.6), some β values may lead to $k_0 < 2$, i.e., loading

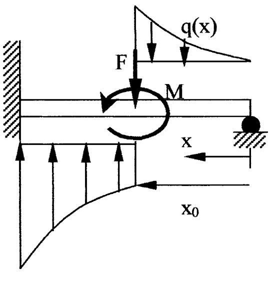

Fig. 6. Loading configuration of Eq. (5.4).

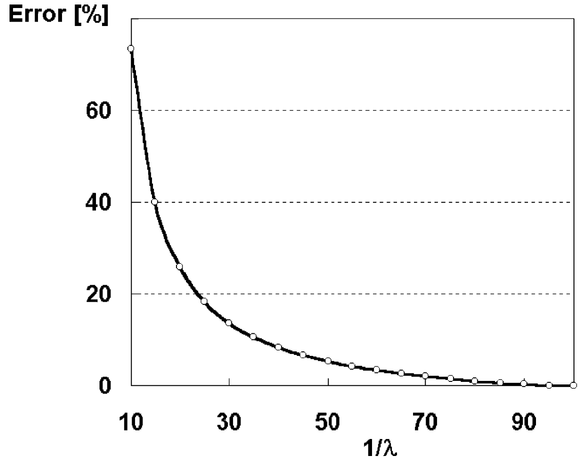

Fig. 7. Relative error of the morphology-strength effect (η), between numerical integration and analytical results, for different correlation lengths λ. Exponential loading shape: $|M_h| = \exp(kx)$, $k = 0.45$, $\beta = 22$.

singularities. Therefore, an additional nonsingular parametric case is examined for comparison ($|M_h| = \exp(kx)$). The external loading distribution $q(x)$ is similar to the one shown in Fig. 6, and $\bar{\eta}_{\max}$ is found by the same procedure. Fig. 5 shows that the two cases (power law and exponential) gave similar results. We therefore conclude, that the bound in (4.24) is very close to the supremum.

Another important issue concerns those loading shapes, which cause the least strength-heterogeneity effect. For demonstration, two additional loading examples (external moment at $x = 0$ and a concentrated force at $x = 0.5$) are also shown in Fig. 5 and show that the heterogeneity effect may also be negligibly small for some loading shapes.

Finally, an accuracy study, which evaluates the various approximations made along the way for the analytical solution, was carried out. A test case for which $|M_h| = \exp(kx)$ was chosen, with $\beta = 22$ and $k = 0.45$, producing a large morphology-strength effect (see Fig. 4). A common two-point probability function was chosen as:

$$\langle S'(x)S'(X)\rangle = \langle S'^2\rangle \exp\left(-\frac{|x - X|}{\lambda/2}\right). \tag{5.7}$$

Note that (5.7) must be written in a form that fits (3.19). Then, the relative morphological effect (η) was calculated by numerical integration of (4.8) for each λ. ΔR_h and $\langle R'^2\rangle$ were calculated directly from (3.15) and (3.21a) without approximations. The ratio $|\eta_{\text{numeric}} - \eta_{\text{analytic}}|/\eta_{\text{numeric}}$, which is the relative deviation (error) from the accurate numerical solution is shown in Fig. 7. It is seen that the error diminishes as λ decreases as expected, which validates the various approximations taken to reach the analytical solution.

6. Conclusion

Some general conclusions from this study are:

1. For the example studied here, the heterogeneity effect on strength was found to be 8% for a relative microsize scale of 1%, showing the importance of the phenomenon for future design of small size heterogeneous structures.
2. It has been shown, that the morphology effect on reliability is proportional to the two-point correlation length (roughly the "grains size"). However, this analytical prediction is valid only for "small grains"

and small statistical variance of the moduli. Expanding the analysis beyond this range is of great importance.
3. Moduli morphology and strength have been considered here as statistically independent material properties, while practically there is definitely a statistical correlation between the two. This is an important subject, which should be studied theoretically and verified experimentally.
4. The interaction between material morphology, loading geometry and the basic strength distribution function is complex and nonlinear even for a case of a simple beam, having one degree of indeterminacy. This may explain the difficulty in finding predictive tools for more general microstructures, and suggests that new approaches should be considered.

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